Brownian motion in an aging medium

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The motion of a particle in an aging medium can be described by the generalized Langevin equation, in the limit of long waiting time t_w where the medium is in a quasistationary regime at the scale of the observation times investigated ($t \ll t_w$). In this framework, we analyze the link between the Brownian motion and the effective temperature which characterizes the out-of-equilibrium properties of the medium. This effective temperature involves a frequency-dependent effective temperature $\mathcal{T}_{\mathrm{eff}}(\omega)$ formally identical to a generalized susceptibility. The analytical results are reported in the case when $\mathcal{T}_{\mathrm{eff}}(\omega)$ is mapped to the universal non-Debye power-law ac response met for instance in dielectrics. In the particular case where the viscous friction coefficient is a power law $\gamma(\omega) \propto |\omega|^{\delta-1}$, contact is made with the heuristic expression $T_{\mathrm{eff}}=T[1+(\omega/\omega_0)^{\alpha}]$, postulated in prior experimental and theoretical works. A closed analytic form of the time correlation function of the medium coordinate (the noise force) $C_{FF}(t-t')=\langle F(t)F(t')\rangle$ is obtained, in the subdiffusive regime ($\delta < 1$) where $C_{FF}(t-t')$ is a regular function. This time correlation is long range. We also determine another effective temperature $T'_{\mathrm{eff}}(t-t')$ of the medium, usually defined in aging systems as the temperature associated with the violation of the fluctuation-dissipation theorem in its time formulation. This temperature takes the form $T'_{\mathrm{eff}}(t-t')=T[1+(|t-t'|/t_0)^{-\alpha}]>T$. The results are discussed and compared with experiments.

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I. INTRODUCTION

Statistical properties of out-of-equilibrium systems are characterized by the violation of equilibrium theorems such as the fluctuation-dissipation theorem (FDT). Since the relation between the autocorrelation $C(t,t')$ and the susceptibility $\chi(t,t')$ of any variable at equilibrium in the classical limit of the FDT is linear in temperature, the violation factor of the FDT for an aging variable is commonly used to define an effective temperature T'_{eff} by the relation [1]

$$\theta(t-t')\frac{\partial C(t,t')}{\partial t'} = k_B T'_{\mathrm{eff}}(t,t')\chi(t,t'). \qquad (1.1)$$

The Brownian motion of a free particle is a simple example of an aging system [2]. This motion is described by a generalized Langevin equation [3,4], in which the effect of the environment (the thermal bath) is encoded in a friction term and a noise term. Under the effect of the diffusion of the particle, its displacement is an aging variable, and a quantitative study of the related effective temperature has been achieved when the bath is at equilibrium [2,5].

The case when the bath is itself out of equilibrium is much more intricate. This situation is met in experiments where the Brownian diffusing particle is a tracer, the motion of which is used as a probe to investigate the out-of-equilibrium properties of the aging bath in which it is immersed. One difficulty comes from the experiments themselves. While the formalism of aging phenomena is best done in real time-space, the use of optical spectroscopy to investigate out-of-equilibrium experimental properties makes desirable a different theoretical approach of aging in the Fourier space of frequencies ω. Hohenberg and Shraiman [6] have defined an effective temperature $T_{\mathrm{eff}}(\omega)$ for stationary nonequilibrium systems through the relation between fluctuation and dissipation in the Fourier space:

$$\omega C(\omega) = 2k_B T_{\mathrm{eff}}(\omega)\mathrm{Im}\,\chi(\omega). \qquad (1.2)$$

At equilibrium, $T'_{\mathrm{eff}}(t,t')=T_{\mathrm{eff}}(\omega)=T$, and Eqs. (1.1) and (1.2) reduce to the FDT expressed in real and Fourier space, respectively. Note Eq. (1.1) depends on two times t,t', while Eq. (1.2) depends on one frequency only, because of the stationary process corresponding to the case when all quantities in Eq. (1.1) depend on $t-t'$ only. We only consider this case in the present work. In practice, this situation can be met in aging sustems, when "observed" at a time scale ω^{-1} short with respect to the waiting time t_w. In this case, the susceptibility, the correlation function, and the effective temperature defined from Eq. (1.2) are parametrized by the waiting time or age t_w [7,8]. The fluctuation and dissipation of the medium are studied at given age t_w, as a function of ω, in a frequency range that satisfies $\omega^{-1} \ll t_w$.

Note that aging is usually defined as the lack of invariance by time translation in the infinite time limit. Aging itself is actually beyond the scope of this study, since the use of Eq. (1.2) requires stationary dynamics. Nevertheless, the term aging medium is used, because the observation of stationary out-of-equilibrium dynamics on a large time scale requires systems with slow dynamics, a feature of aging systems.

Recently, several experimental determinations of effective temperature in the Fourier space have been achieved in various aging systems [9–12]. In a first attempt to address this problem, the generalized Langevin equation has been used to determine the motion of the diffusing particle embedded in

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the aging system [13]. It has been argued in this work that the asymptotic Fourier analysis (corresponding to the long-time limit) can be used to characterize the aging medium, from the measurement of the mobility and the displacement autocorrelation function of the Brownian particle. However, equivalent expressions of the FDT lead to nonequivalent effective temperatures when the FDT is violated [13,14]. The first purpose of this papers is to investigate the self-consistency of this approach, and analyze the link with experimental studies of the Brownian motion.

In case Eq. (1.2) cannot be used (time invariance broken), the study of the dynamics in the real time-space from Eq. (1.1) is unavoidable. We then found it also desirable to study $T'_{\mathrm{eff}}(t,t')$ for the aging medium in our model although we only consider the stationary case: this is the only temperature that can be investigated in the range of observation and waiting times where a frequency-dependent effective temperature is not defined. It is the second purpose of the present work to report this study.

The paper is organized as follows. We briefly recall the Langevin equation used to describe the diffusion of the Brownian particle (Sec. II). We recall in Sec. III the formulations of the fluctuation-dissipation theorems which apply to both the velocity of the Brownian particle (first FDT) and the random force of the thermal bath (second FDT), provided the bath is at equilibrium. When the bath is not at equilibrium, its aging can be characterized by an effective temperature in the frequency space, which has the analytic properties of a generalized susceptibility, so that universal laws of response functions can be used to model it. We envision in Sec. IV the case of the universal power-law ac response applying to a wide class of problems and materials, including dielectrics and ferroelectrics, in a broad range of frequencies. To make contact with prior works, we consider the case when the time or frequency dependence of the friction reduces to a power law. The link with recent experiments [12] is then discussed. We analyze in Sec. V the correlation function of the aging variable of the medium under study, namely, the friction noise. We derive the corresponding effective time-dependent temperature T'_{eff} which describes the aging of the system under study, as defined from the violation of the real time formulation of the second FDT. We find that it satisfies the criterion $T'_{\mathrm{eff}}(\tau) > T$ for any observation time τ, as is required for this parameter to have the meaning of an effective temperature [1,7,8].

II. DIFFUSION IN A STATIONARY MEDIUM

The motion of a diffusing particle of mass m evolving in a stationary medium is usually described by the generalized Langevin equation [3,4]

$$m\frac{dv}{dt} = -m\int_{-\infty}^{\infty} \tilde{\gamma}(t-t')v(t')dt' + F(t), \quad v = \frac{dx}{dt}, \quad (2.1)$$

in which $F(t)$ is the Langevin random force acting on the particle and $\tilde{\gamma}(t)=\gamma(t)\theta(t)$ is a retarded friction kernel. The Fourier transform of Eq. (2.1) is

$$v(\omega) = \mu(\omega)F(\omega), \quad \mu(\omega) = \frac{1}{m[\gamma(\omega) - i\omega]}, \quad (2.2)$$

which defines the frequency-dependent mobility μ of the particle. Both $F(t)$ and the solution $v(t)$ of the generalized Langevin equation (2.1) can then be viewed as stationary random processes with spectral densities linked by

$$C_{vv}(\omega) = |\mu(\omega)|^2 C_{FF}(\omega). \quad (2.3)$$

In Eq. (2.2) the friction coefficient $\gamma(\omega)$ is the Fourier transform of the retarded kernel $\tilde{\gamma}(t)$, as defined by $\gamma(\omega) = \int_{-\infty}^{\infty}\tilde{\gamma}(t)e^{i\omega t}dt$.

III. CASE OF A BATH AT EQUILIBRIUM: FLUCTUATION-DISSIPATION THEOREMS

We start from a prior work [13], and use the same notations. In particular, the expressions of the first FDT which applies to the velocity of the Brownian particle and the second FDT relative to the bath have already been derived in [13], and we just recall here the results [Eqs. (2.9)–(2.14) in Ref. 13]. For the first FDT,

$$\begin{aligned}\int_0^{\infty} \langle v(t)v(0)\rangle e^{i\omega t}dt &= kT\mu(\omega) \\ &= -\frac{\omega^2}{2}\int_0^{\infty}\langle [x(t)-x(0)]^2\rangle e^{i\omega t}dt, \end{aligned} \quad (3.1)$$

$$C_{vv}(\omega) = \int_{-\infty}^{\infty}\langle v(t)v(0)\rangle e^{i\omega t}dt = kT2\,\mathrm{Re}\,\mu(\omega). \quad (3.2)$$

We have added in Eq. (3.1) the expression of the mobility in terms of the mean square displacement $\Delta x^2(t)=\langle [x(t)-x(0)]^2\rangle$ displayed in Ref. 15. This expression is more conveniently used to make contact with experiments, although it is equivalent to the expression of the mobility as a function of $\langle v(t)v(0)\rangle$ displayed on the left side of Eq. (3.1). We will return to this point in Sec. IV. For the second FDT,

$$\int_0^{\infty}\langle F(t)F(0)\rangle e^{i\omega t}dt = mk_BT\gamma(\omega), \quad (3.3)$$

$$C_{FF}(\omega) = \int_{-\infty}^{\infty}\langle F(t)F(0)\rangle e^{i\omega t}dt = mk_BT2\,\mathrm{Re}\,\gamma(\omega). \quad (3.4)$$

Equations (3.1) and (3.2) on one hand, and (3.3) and (3.4) on the other hand, are equivalent (see Appendix A), and they correspond to the Einstein-Kubo formula and the Nyquist formula, respectively. However, this equivalence, which holds at equilibrium, cannot be maintained out of equilibrium, when one attempts to extend the FDT's with the help of a frequency-dependent effective temperature [7,8].

IV. CASE OF A BATH OUT-OF-EQUILIBRIUM MEDIUM: EFFECTIVE TEMPERATURES

Let us now consider the case where the Brownian particle is immersed in an out-of-equilibrium medium, so that neither the Einstein nor the Nyquist formulas hold true. The concept of effective temperature has been introduced to quantify the violation of these formulas and characterize the evolution of the aging variables. In this situation, the system under study is the aging medium itself, so that we are primarily interested in the effective temperature associated with the violation of the FDT relative to the bath, i.e., the violation of Eqs. (3.3) and (3.4). However, as noticed in Ref. 13, the effective temperatures defined from these two equations are not the same.

A. Definition of the effective temperature

Let us start from Eq. (3.3), and then define an effective temperature $\mathcal{T}_{\rm eff}(\omega)$ by the substitution

$$T\gamma(\omega) \to \mathcal{T}_{\rm eff}(\omega)\gamma(\omega) \tag{4.1}$$

so that

$$\int_0^\infty \langle F(t)F\rangle e^{i\omega t}dt = mk_B\mathcal{T}_{\rm eff}(\omega)\gamma(\omega). \tag{4.2}$$

$\mathcal{T}_{\rm eff}(\omega)$ must have the same analytic properties as $\gamma(\omega)$, i.e., be analytic in the upper complex plane, so that it can be viewed as an effective ac susceptibility. We can then make use of this mapping between this effective temperature and the ac susceptibility to model $\mathcal{T}_{\rm eff}$. For instance, let us consider the case when this response is a power law [16–18], as this is a universal dielectric response law that describes the non-Debye response $\chi(\omega)$ of a wide class of dielectrics and ferroelectrics in a broad range of frequencies. The substitution $\chi(\omega)\to\mathcal{T}_{\rm eff}(\omega)$, $\chi_\infty\to T$ in this power-law response, for instance in Eqs. (1) and (2) of Ref. 18, gives, respectively,

$$\mathcal{T}_{\rm eff}(\omega) = T[1+(-i\omega\tau_0)^\alpha], \quad \alpha<0, \tag{4.3}$$

$$\mathrm{Im}\,\mathcal{T}_{\rm eff}(\omega) = -\,\mathrm{sgn}(\omega)[\mathrm{Re}\,\mathcal{T}_{\rm eff}(\omega)-T]\tan\pi\alpha/2, \tag{4.4}$$

where α is a negative constant [19]. In this mapping, as χ_∞ is the high-frequency limit of the dielectric response ($\alpha<0$), T is the high-frequency limit of the effective temperature. In order words, T is defined as the temperature at which the relaxing (nonaging) high-frequency modes "thermalize." τ_0 is the characteristic relaxation time. Equation (4.4) written for real ω follows from the Kramers-Kronig relations linking the real and imaginary parts of the susceptibilitylike effective temperature $\mathcal{T}_{\rm eff}$.

B. Link with the Brownian motion

The study of the Brownian motion does not give direct access to $T_{\rm eff}$. Instead, we have argued in Ref. 13 that it gives access to another effective temperature $T_{\rm eff}(\omega)$ defined according to Eq. (1.2). Applied to our problem, this equation amounts to making the substitution

$$T\,\mathrm{Re}\,\gamma(\omega) \to T_{\rm eff}(\omega)\mathrm{Re}\,\gamma(\omega) \tag{4.5}$$

into Eq. (3.4), so that

$$C_{FF}(\omega) = 2mk_BT_{\rm eff}(\omega)\mathrm{Re}\,\gamma(\omega). \tag{4.6}$$

As noticed in our prior work [see Eq. (4.14) in Ref. 13], $T_{\rm eff}\neq\mathcal{T}_{\rm eff}$. After Eqs. (4.1) and (4.5) (see also Appendix A), these effective temperatures are linked by the relation [14]

$$T_{\rm eff}(\omega)\mathrm{Re}\,\gamma(\omega) = \mathrm{Re}[\gamma(\omega)\mathcal{T}_{\rm eff}(\omega)]. \tag{4.7}$$

The solution of this equation requires a model for the friction. We assume that the friction takes the same generic form as in the case where the bath is made of a continuum of harmonic oscillators [20]:

$$\gamma(t) = \frac{2}{\pi}\gamma_\delta\int_0^\infty d\omega\left(\frac{\omega}{\tilde\omega}\right)^{\delta-1}f_c(\omega)\cos\omega t. \tag{4.8}$$

γ_δ is a constant with the dimension of a frequency, and determines the strength of the friction. The spectral density of low-frequency modes of the environmental coupling is assumed to be a power law $J(\omega)\propto\omega^\delta$, which defines the parameter δ. fc is a cutoff for the spectral density in the high-frequency limit. In the case of interest where $0<\delta<2$ (when $\delta>2$, the Brownian particle acts as a free particle with renormalized mass) the choice of a Lorentzian cutoff

$$f_c = \omega_c^2/(\omega_c^2+\omega^2)$$

is sufficient to avoid any divergence associated with the upper bound ∞ of the integrals on the variable ω. The analytic expression of $\gamma(\omega)$ with such a cutoff has been reported elsewhere [5,20]. When $\omega\ll\omega_c$, we can take the limit $\omega_c\to\infty$ in Eqs. (6) and (10) of Ref. 5 which reduce to [21]

$$\gamma(\omega) = \omega_\delta\left(\frac{-i\omega}{\omega_\delta}\right)^{\delta-1}, \quad |\omega|\ll\omega_c, \tag{4.9}$$

where

$$\omega_\delta^{2-\delta} = \gamma_\delta\frac{1}{\tilde\omega^{\delta-1}}\frac{1}{\sin(\delta\pi/2)}. \tag{4.10}$$

Substituting into Eq. (4.7) the expressions of $\mathcal{T}_{\rm eff}(\omega)$ and $\gamma(\omega)$ displayed in Eqs. (4.3) and (4.9), we find $T_{\rm eff}$ under the form

$$T_{\rm eff}(\omega) = T\left[1+\left(\frac{|\omega|}{\omega_0}\right)^\alpha\right], \quad \alpha<0, \quad |\omega|\ll\omega_c, \tag{4.11}$$

with $\omega_0\propto\tau_0^{-1}$. To make contact with prior works [11–14], we choose to keep ω_0 as the parameter of the model, and express τ_0 as a function of ω_0, which amounts to writing Eq. (4.3) under the form

$$\mathcal{T}_{\rm eff}(\omega) = T\left[1+\frac{\sin(\delta\pi/2)}{\sin[(\delta+\alpha)\pi/2]}\left(\frac{-i\omega}{\omega_0}\right)^\alpha\right], \quad \alpha<0, \quad |\omega|\ll\omega_c. \tag{4.12}$$

The frequency ω_0 separates low frequencies (i.e., slow modes responsible for the aging of the medium), for which one has $T_{\rm eff}(\omega)\sim T(|\omega|/\omega_0)^\alpha$, from high frequencies (i.e., fast modes), for which one has $T_{\rm eff}(\omega)\sim T$. The equilibrium situation would correspond to $\omega_0=0$, in which case $T_{\rm eff}(\omega)=T$

for any ω. It can also formally be retrieved by taking the limit $\alpha \to 0^-$ associated with the substitution $T \to T/2$. Actually, Eq. (4.7) gives $T_{\rm eff}$ as a function of $\gamma(\omega)$ and $\mathcal{T}_{\rm eff}(\omega)$, and has been used in this way in the present work. However, the opposite is true, i.e., if $\gamma(\omega)$ is given by Eq. (4.9) and $T_{\rm eff}$ is given by Eq. (4.11), then Eq. (4.7) is an equation in $\mathcal{T}_{\rm eff}(\omega)$ which has Eq. (4.12) as the unique solution. This is shown in Appendix B. Owing to the power law for the friction, these two effective temperatures satisfy the same power law of the form $[1+(\omega/\omega_0')^\alpha]$ with the same α. They differ only by a renormalization of the parameter ω_0', eventually complex. It is important, however, to note that this same generic power law for both $T_{\rm eff}(\omega)$ and $\mathcal{T}_{\rm eff}(\omega)$ only results from the very particular choice of the power law in Eq. (4.9) for the friction $\gamma(\omega)$. Should the friction be different, or at high frequency (ω not small with respect to ω_c), then the $T_{\rm eff}(\omega)$ solution of Eq. (4.7) would have a different functional dependence. Nevertheless, this particular case is of interest, since Eq. (4.11) has been used as a heuristic law postulated both in experiments [11] to analyze the data, and in prior theoretical works [13,14] to model the out-of-equilibrium dynamics.

C. Link with experiments

On an experimental point of view, the motion of the Brownian particle can be investigated, aiming to determine the effective temperature that characterizes the aging medium [12]. This motion is linked to the medium through Eq. (2.3), with the mobility μ given in Eq. (2.2). Combined with Eq. (4.6), these equations (2.2) and (2.3) lead to

$$C_{vv}(\omega) = k_B T_{\rm eff}(\omega) 2\,\mathrm{Re}\,\mu(\omega). \qquad (4.13)$$

$T_{\rm eff}$ is then not only an effective temperature associated with the bath, after Eqs. (4.6) and (4.7), but also an effective temperature for the Brownian particle, associated with the violations of the FDT expressed in the form of Eqs. (3.2) and (3.4), respectively. In principle, this temperature can be deduced from independent measurements of $\mathrm{Re}\,\mu(\omega)$ and $C_{vv}(\omega)$. On an experimental point of view, however, it is easier to determine the mean square displacement of the particle than its velocity correlation function. Therefore, we had better consider the effective temperature $\mathcal{T}_{\rm eff}^{(1)}$ associated with the violation of the FDT expressed in the form of Eq. (3.2):

$$\mu(\omega) k_B \mathcal{T}_{\rm eff}^{(1)}(\omega) = \int_0^\infty \langle v(t) v\rangle e^{i\omega t} dt = -\frac{\omega^2}{2}\int_0^\infty \Delta x^2(t) e^{i\omega t} dt. \qquad (4.14)$$

$\mathcal{T}_{\rm eff}^{(1)}$ is defined from Eqs. (4.13) and (4.14) as is shown in Appendix A. The result, already derived in Ref. [14], is:

$$T_{\rm eff}(\omega)\mathrm{Re}\,\mu(\omega) = \mathrm{Re}[\mu(\omega) k \mathcal{T}_{\rm eff}^{(1)}(\omega)]. \qquad (4.15)$$

Comparing Eqs. (4.7) and (4.15), we find that the equations differ only by the substitution $\gamma(\omega) \to \mu(\omega)$. This is, however, an important difference, since μ, in contrast to γ, is not a power law of ω, unless the term $-i\omega$ is negligible with respect to $\gamma(\omega)$ in the expression $\mu(\omega) = m[\gamma(\omega) - i\omega]^{-1}$ given by Eq. (2.2). This situation is met only in the frequency range $\omega \ll \omega_\delta$, which corresponds to the long-time limit $\omega_\delta t \gg 1$, in which inertia can be neglected.

1. The long-time limit

In this limit,

$$\mu(\omega) \simeq \frac{1}{m\gamma(\omega)}, \quad |\omega| \ll \omega_\delta. \qquad (4.16)$$

In this case, the change in the exponent between $\gamma \propto \omega^{\delta-1}$ and $\mu \propto \omega^{1-\delta}$ is pictured by $1-\delta \to \delta - 1$, that is, $\delta \to 2-\delta$, so that $\mathcal{T}_{\rm eff}^{(1)}(\omega)[\alpha,\delta] = \mathcal{T}_{\rm eff}(\omega)[\alpha, 2-\delta]$; hence

$$\mathcal{T}_{\rm eff}^{(1)}(\omega) = T\left[1 + \frac{\sin(\delta\pi/2)}{\sin[(\delta-\alpha)\pi/2]}\left(\frac{-i\omega}{\omega_0}\right)^\alpha\right], \quad |\omega| \ll \omega_\delta. \qquad (4.17)$$

Note that $\omega_\delta \ll \omega_c$. In practice, we could even take the limit $\omega_c \to \infty$ to study the Brownian motion with a bath at equilibrium [5]. The condition $|\omega| \ll \omega_\delta$ is thus much more stringent than the condition $|\omega| \ll \omega_c$ in Eq. (4.12), and actually it corresponds to the long-time limit already investigated in prior work [13] since $|\omega| \ll \omega_\delta$ means $t \gg \omega_\delta^{-1}$. In particular, we have shown [Eq. (5.8) in Ref. [13]] that at such long times

$$\Delta x^2(t) \simeq \frac{2k_B T}{m}\omega_\delta^{\delta-2}\omega_0^{-\alpha}\frac{1}{\Gamma(\delta-\alpha+1)}\frac{\sin\delta\pi/2}{\sin(\delta-\alpha)\pi/2}t^{\delta-\alpha},$$
$$\omega_\delta t \gg 1. \qquad (4.18)$$

Experiments made in this long-time limit have recently confirmed this time power law of the mean square displacement $\Delta x^2(t)$, and the frequency power law of the mobility $\mu(\omega)$ [12]. The authors of this work could deduce $\mathcal{T}_{\rm eff}^{(1)}(\omega)$ from their data, using a procedure which can be made more simple. The procedure in [12] is the following. (a) Laplace transform $\Delta x^2(t)$. (b) Derive the Laplace transform $\hat{\mu}(z) = \mu(\omega = iz)$ from the analytic expression of $\mu(\omega)$ by analytic continuation. (c) Determine the Laplace transform $\hat{\mathcal{T}}_{\rm eff}^{(1)}(z)$ from the analytic continuation of Eq. (4.14) with $\omega = iz$, namely, $2\hat{\mu}(z) k_B \hat{\mathcal{T}}_{\rm eff}^{(1)}(z) = z^2 \widehat{\Delta x^2}(z)$. (d) Substitute z by $-i\omega$ to obtain finally $\mathcal{T}_{\rm eff}^{(1)}(\omega)$. In the case where $\mu(\omega)$ and $\Delta x^2(t)$ follow a power law, a simpler analysis could be the following. (a) Identify the power law $\mu(\omega)$ with Eqs. (4.16) and (4.9) to deduce ω_δ and δ. (b) Identify $\Delta x^2(t)$ with Eq. (4.18) to derive α and ω_0. (c) Substitute in Eq. (4.17) to deduce $\mathcal{T}_{\rm eff}^{(1)}(\omega)$. In addition, since the experiments aim at the determination of the properties of the medium in which the particle is immersed, the numerical application should rather be made in Eqs. (4.11) and (4.12) to obtain the effective temperature of the bath, rather than in Eq. (4.17) which gives the effective temperature of the particle. This simple situation has been met in experiments reported in Ref. [12]. The fact that only the second term in the brackets of Eq. (4.17) has been detected [12] is simply due to the fact that the first one is negligible in this long-time limit, since $\omega/\omega_0 \gg 1$ and $\alpha < 0$. Note, however, that $\mu(\omega)$ has been found in [12] to reduce to a power law only in a finite range of waiting times where the phase of $\mu(\omega)$ does not depend significantly on ω [12]. A significant departure from this behavior has been evidenced at long waiting times, in which case $\mu(\omega)$ is no longer a power law.

2. The general case

Most often, the measurements of $\mu(\omega)$ will reveal that it does not follow a power law, either because the inertial term is not negligible, i.e., we are not in the long-time limit, or because the analytic form of $\mu(\omega)$ does not reduce to Eqs. (2.2) and (4.9). The latter case will be met, for instance, if the medium under study is a disordered solid [15]. Still, one can readily determine the effective temperature of the medium from measurements of the mean square displacement and mobility of the Brownian particle, using the following procedure. (a) Use interpolation techniques (such as spline methods for instance) to determine $\Delta x^2(t)$ at any time from data at discrete times. (b) Fast Fourier transform techniques can be used to compute the partial Fourier transform in the second member of Eq. (4.14):

$$k_B\mathcal{T}^{(1)}_{\text{eff}}(\omega) = -\frac{\omega^2}{2\mu(\omega)}\int_0^\infty \Delta x^2(t)e^{i\omega t}dt \tag{4.19}$$

at any ω, provided that $\Delta x^2(t)$ diverges more smoothly than t^2 in the long-time limit. This last condition simply means that the anomalous diffusion exponent ν must be smaller than 2 to avoid divergence associated with kinematical effects. (c) Use Eq. (4.19) to determine $\mathcal{T}^{(1)}_{\text{eff}}(\omega)$ from the data $\mu(\omega)$ and the result of the integration. (d) T_{eff} can be determined from Eq. (4.15). This simple procedure allows one to derive $T_{\text{eff}}(\omega)$ from the measurements of the mobility and the mean square displacement of the Brownian particle even if the analytic form of $\mu(\omega)$ is unknown. This is a proof that the analysis of the motion of the Brownian particle is a powerful tool to determine the effective temperature of an aging medium in which the particle is embedded, provided the system is in a stationary regime.

We also note that the use of the procedure involving the Laplace transform suggested in some earlier works [12,14] would be even more intricate than in the long-time limit, since the switch from $\mu(\omega)$ to $\hat{\mu}(z)$ cannot be done if the analytic form of $\mu(\omega)$ is unknown. One can always use a fitting procedure to approximate the experimental curves by some polynomial or another more or less sophisticated function, but this can only introduce a source of error in the analysis of the data and an additional step in the analysis process. That is why, in our view, the Fourier transform analysis of Eq. (4.19) following Sher and Lax in a generalized theory of the mobility for stochastic transport [15] should be preferred to the Laplace transform analysis of the same equation with $\omega = iz$ envisioned in [12,14].

On the other hand, the use of the Laplace transform is suited to the study of the velocity correlation function when the bath is at equilibrium [21]. Let us show, however, that this is no longer the case when the bath is out of equilibrium. The fact that the mobility reduces to a power law only in the long-time limit means that the velocity correlation function at shorter times does not reduce to a simple analytic form. The Laplace Fourier transform of Eq. (2.1) is

$$z\hat{v}(z) + \hat{\gamma}(z)\hat{v}(z) = \hat{F}(z)/m + v_0, \tag{4.20}$$

where $v_0 = v(t=0)$; hence

$$\langle\hat{v}(z)\rangle = \frac{v_0}{z + \hat{\gamma}(z)}. \tag{4.21}$$

After Eq. (4.9),

$$\gamma(i\omega) = \hat{\gamma}(z) = \omega_\delta^{2-\delta}z^{\delta-1}, \tag{4.22}$$

and the inverse Laplace Fourier transform of Eq. (4.21) is [21]

$$\langle v(t)\rangle = v(t=0)E_{2-\delta}(-(\omega_\delta t)^{2-\delta}), \quad \omega_c t \gg 1, \tag{4.23}$$

where $E_{2-\delta}(x)$ is the Mittag-Leffler function of index $2-\delta$. The condition $\omega_c t \gg 1$ is due to the fact that Eq. (4.9), from which Eq. (4.23) issues, is valid only in the range $\omega \ll \omega_c$. On the other hand, the Laplace transform of the velocity correlation function is, after Eq. (4.14),

$$\hat{C}_{vv}(z) = k_B\hat{\mu}(z)\hat{\mathcal{T}}^1_{\text{eff}}(z). \tag{4.24}$$

When the medium in which the particle is embedded is at equilibrium, $\mathcal{T}^1_{\text{eff}} = T$. In this case, after Eqs. (4.21) and (4.22), and after the analytic continuation of Eq. (2.2) for $z = i\omega$: $\mu(z) = [z + \hat{\gamma}(z)]^{-1}$, we find $C_{vv}(t) = (k_BT/mv_0)\langle v(t)\rangle$. This is the regression theorem, which states that the correlation function and the mean value of a variable at equilibrium follow the same law [13]. When the medium is aging, however, the velocity of the Brownian particle is also out of equilibrium, and this regression theorem is violated. To be more specific, Eq. (4.21) and Eq. (4.23) still hold true, but $\hat{\mathcal{T}}^1_{\text{eff}}$ in Eq. (4.24) depends on z so that $C_{vv}(t)$ is no longer proportional to $\langle v(t)\rangle$. Due to the breakdown of the regression theorem, the velocity correlation function can no longer by expressed as a Mittag-Leffler function. Only its asymptotic behavior at long times is known (and reported in Ref. [13]), corresponding to the case where Eqs. (4.17) and (4.18) can be used. At shorter times, Eq. (4.19) is thus definitely of a most practical use to determine the effective temperature $\mathcal{T}^1_{\text{eff}}$, not only from an experimental point of view, but also from a theoretical point of view.

Let us recall that the last step (d), use Eq. (4.15) to determine $T_{\text{eff}}(\omega)$, sometimes missing in prior experimental works, is of basic importance here, since the functional dependences of $\mathcal{T}^1_{\text{eff}}(\omega)$ and $T_{\text{eff}}(\omega)$ are not the same in the general case.

V. RANDOM FORCE FLUCTUATION AND DISSIPATION

Equation (4.6) for $C_{FF}(\omega)$, with $\gamma(\omega)$ given by Eq. (4.9) and $T_{\text{eff}}(\omega)$ given by Eq. (4.11), yields the noise spectral density:

$$C_{FF}(\omega) = mk_B T\left[1 + \left(\frac{|\omega|}{\omega_0}\right)^{\alpha}\right] 2\omega_{\delta}\left(\frac{|\omega|}{\omega_{\delta}}\right)^{\delta-1} \sin\frac{\delta\pi}{2},$$
$$|\omega| \ll \omega_c. \tag{5.1}$$

Note the second term in the square brackets is the leading term at small frequencies. Therefore, the chosen modeling of the effective temperature ensures that the density of slow modes in the noise is larger than in a thermal bath at temperature T [7,13,22]. This is indeed expected, since the slow modes are responsible for the aging. The random force correlation function $\langle F(t)F(t')\rangle$ can be obtained by means of the Fourier integral (5.1), which converges provided that the condition[1]

$$\alpha + \delta > 0 \tag{5.2}$$

is satisfied. Note this is equivalent to the condition

$$\mathcal{T}_{\mathrm{eff}}(\omega) > T \tag{5.3}$$

at any ω, as can be seen from Eq. (4.12). One can use either Laplace transform or Fourier transform to determine the noise correlation function. In continuity with the previous section where we used the Laplace transform to analyze the velocity correlation function, we follow here the same procedure for the noise. The Laplace transform of C_{FF} is readily deduced from the analytic continuation of Eq. (4.2) which, for $\omega = iz$, can be written

$$\hat{C}_{FF}(z) = mk_B \hat{\mathcal{T}}_{\mathrm{eff}}(z)\hat{\gamma}(z). \tag{5.4}$$

After Eq. (4.9) and (4.22) and Eq. (4.12), Eq. (5.4) can be written

$$\hat{C}_{FF}(z) = mk_B T\omega_{\delta}\left[\frac{z^{\delta-1}}{\omega_{\delta}^{\delta-1}} + \frac{\sin(\delta\pi/2)}{\sin[(\delta+\alpha)\pi/2]}\frac{z^{\delta+\alpha-1}}{\omega_{\delta}^{\delta-1}\omega_0^{\alpha}}\right]. \tag{5.5}$$

The inverse Laplace transform of this power-law series is a regular function if and only if $\hat{C}_{FF}(z) \to 0$ when $z \to \infty$, i.e., if $\delta < 1$. In this case, the result is

$$\langle F(t)F(t')\rangle = mk_B T\gamma(|t-t'|) + \frac{2}{\pi} mk_B T\omega_{\delta}^2 \sin\frac{\pi\delta}{2}\cos\frac{\pi(\alpha+\delta)}{2} \times \Gamma(\alpha+\delta)\frac{|t-t'|^{-(\delta+\alpha)}}{\omega_{\delta}^{\delta}\omega_0^{\alpha}}, \quad \delta < 1, \tag{5.6}$$

with

$$\gamma(t) = \frac{2}{\pi}\omega_{\delta}^2 \sin\frac{\pi\delta}{2}\cos\frac{\pi\delta}{2}\Gamma(\delta)\frac{|t|^{-\delta}}{\omega_{\delta}^{\delta}}, \quad \delta < 1. \tag{5.7}$$

The first term in Eq. (5.6) is the inverse Fourier transform of Eq. (3.4) so the second term is the correction to the second FDT induced by the aging of the medium under study.

[1]The condition (5.2) is a low-frequency criterion: it stems from the low-ω behavior of the integrand. As for the convergence at infinity, it is effective, due to the existence of the cutoff at ω_c.

The Ohmic case $\delta = 1$ is the limit case where the inverse Laplace transform is not a function, but a distribution:

$$\langle F(t)F(t')\rangle = \gamma_1 mk_B T\left[2\delta(t-t') + \frac{2}{\pi}\cos\frac{\pi(\alpha+1)}{2} \times \Gamma(\alpha+1)\frac{|t-t'|^{-(\alpha+1)}}{\omega_0^{\alpha}}\right]. \tag{5.8}$$

It is important to note that Eqs. (5.6)–(5.8) are inverse Fourier transforms of expressions deduced from analytic continuation ($\omega = iz$) of functions established for real ω in the limit $\omega \ll \omega_c$. Therefore, Eqs. (5.6)–(5.8) are valid, like Eq. (4.23), only in the limit $\omega_c t \gg 1$. Due to the lack of any singularity in the correlation functions, we can extend Eqs. (5.6) and (5.7) to any time, which amounts to considering the limit $\omega_c \to \infty$. This is also usually done for $\delta = 1$. Indeed, in the limit $\omega_c \to \infty$ at $\delta = 1$, $\gamma(t) \propto \delta(t)$ so that the the limit $\omega_c \to \infty$ amounts to reducing the Langevin equation to its nonretarded form $m dv/dt + \eta v(t) = F(t)$, with η the viscosity. The Dirac term in Eq. (5.8) corresponds to the white Gaussian noise of zero mean $\langle F(t)F(t')\rangle = 2\eta k_B T\delta(t)$ characteristic of the Brownian motion in the Ohmic case at equilibrium.

When $\delta > 1$, however, the cutoff at ω_c is needed to avoid divergence of the Fourier (or Laplace) transforms or the integrals, and $\langle F(t)F(t')\rangle$ has to be determined by taking the inverse Fourier transform of Eq. (4.6), or the inverse Laplace transform from Eq. (5.4) with $\gamma(\omega)$ or $\hat{\gamma}(z)$ deduced from Eq. (4.8) including the function f_c. The expressions of $\gamma(\omega)$ and $\hat{\gamma}(z)$ are still analytic for the particular case of a Lorentzian cutoff function [20] so that the inverse transform of Eqs. (4.6) and (5.4) can be computed if desired. Note also that the cutoff at high frequency can be skipped for the investigation of the velocity or the displacement correlation function of the particle in the whole range $0 < \delta < 2$, while it cannot be skipped for the noise friction correlation function when $\delta > 1$. This is related to the fact that the noise is much less regular than the correlation functions of the variables of the Brownian particle. The Dirac peak in Eq. (5.8) when $\langle v(t)v(t')\rangle$ reduces to a regular Mittag-Leffler function at equilibrium when $\delta = 1$ is an illustration of this difference. Below, we consider only the sub-Ohmic case $\delta < 1$, since the results in the superdiffusive case depend on the cutoff function f_c, i.e., on the particular medium under study. The case $\delta = 1$ will be discussed separately in the next section, since it is a common case met in experiments, underlying the physics of Brownian motors for instance.

Usually, for aging systems, the effective temperature is associated with the violation of the FDT formulated in real time. In the present case, this effective temperature $T'_{\mathrm{eff}}(t-t')$ should then be defined by Eq. (1.1) which takes the form

$$\chi_{FF}(t-t') = \beta'_{\mathrm{eff}}(t-t')\theta(t-t')\frac{\partial\langle F(t)F(t')\rangle}{\partial t'} \tag{5.9}$$

where $\beta' = 1/(k_B T'_{\mathrm{eff}})$. Note that Eq. (5.9) is written taking into account that we deal with a stationary regime where

$\langle F(t)F(t')\rangle$ and $T'_{\rm eff}$ depend only on $t-t'$ and not on t and t' separately. After Eq. (2.1),

$$\chi_{FF}(t-t') = m\,\theta(t-t')\frac{\partial\gamma(t-t')}{\partial t'}. \tag{5.10}$$

At equilibrium where $T'_{\rm eff}=T$, the identification of the second members of Eqs. (5.9) and (5.10) gives, after integration,

$$\langle F(t)F(t')\rangle = mk_B T\gamma(|t-t'|) \tag{5.11}$$

plus an integration constant which, however, is zero since both members vanish in the limit $|t-t'|\to\infty$. Equation (5.11) is just the inverse Fourier transform of Eq. (3.4). When the medium under study is out of equilibrium, Eqs. (5.6) and (5.7) give, when $t>t'$,

$$\frac{\partial\langle F(t)F(t')\rangle}{\partial t'} = mk_B T\frac{\partial\gamma(t-t')}{\partial t'}\left[1+\frac{\Gamma(\alpha+\delta+1)}{\Gamma(\delta+1)}\times\frac{\cos\pi(\delta+\alpha)/2}{\cos\pi\delta/2}\frac{(t-t')^{-\alpha}}{\omega_0^{\alpha}}\right]. \tag{5.12}$$

From this equation, we can deduce the effective temperature defined by Eq. (5.9):

$$T'_{\rm eff}(t-t') = T\left[1+\frac{\Gamma(\alpha+\delta+1)}{\Gamma(\delta+1)}\frac{\cos\pi(\delta+\alpha)/2}{\cos\pi\delta/2}\frac{(t-t')^{-\alpha}}{\omega_0^{\alpha}}\right],\quad \delta<1. \tag{5.13}$$

Remember the model is meaningful only if $\alpha+2>\delta$, since this condition is equivalent to $\nu<2$ where ν is the anomalous diffusion [13], and equivalent to the condition needed for the velocity correlation function $\langle v(t)v(t')\rangle$ to exist [13]. This condition is actually more stringent than the condition $\alpha+\delta>-2$ for which, after Eq. (5.13)

$$T'_{\rm eff}(t-t') > T. \tag{5.14}$$

Therefore the inequality (5.14) is always satisfied when the model applies. This is known to be the condition for $T'_{\rm eff}$ to be meaningful in terms of the effective temperature for an aging variable [1,8,9].

VI. CONCLUDING REMARKS AND DISCUSSION

The present work analyzes the possibility to use the Brownian motion of a particle as a probe to characterize the aging of the complex medium in which it is immersed. The Brownian motion is described by the Langevin equation in Eq. (2.1), which has been taken in its generalized form including a memory friction. The common route to derive this equation is to model the thermal environment by an infinite and continuous set of harmonic oscillators linearly coupled to the particle. It is commonly accepted that, for a heat bath at *thermal equilibrium*, all the statistic properties of the noise are uniquely fixed, i.e., independent of any further microscopic details of the thermal bath. Then, it has been inferred that any dissipative dynamics of the form (2.1) in contact with an *equilibrium* heat bath can be represented by such a harmonic oscillator bath [23]. Such a harmonic oscillator bath, however, does not age. Therefore, the actual heat bath in the case of an aging medium must be different, and there is no proof that Eq. (2.1) is still relevant to the description of the motion of the particle with such a different bath. In order to justify the use of Eq. (2.1), we have restricted ourselves to the case where the medium, although not at equilibrium, is in a quasistationary state for a given waiting time t_w. The use of the Langevin equation formalism to describe the Brownian motion beyond stationarity is thus questionable. This is a basic reason why the present study is restricted to the stationary case. In addition the concept of a frequency-dependent effective temperature is meaningful only in the stationary case.

The two effective temperatures that are most relevant to describe the evolution of the Brownian particle and the aging medium in the frequency space are $\mathcal{T}^1_{\rm eff}(\omega)$ and $\mathcal{T}_{\rm eff}(\omega)$, respectively. These temperatures are those that have the same expected analytic properties, allowing for their analytic continuation in the complex plane. They equalize only at equilibrium, which can be reached only if the medium is itself at equilibrium. If the medium is aging, $\mathcal{T}^1_{\rm eff}\neq\mathcal{T}_{\rm eff}$ at any frequency. There is, however, another temperature $T_{\rm eff}$ which makes the link between the Brownian motion and the dynamical properties of the aging medium. Measurements of the mobility and mean square displacement of the Brownian particle give direct access to $\mathcal{T}^1_{\rm eff}(\omega)$. Then $T_{\rm eff}$ can be deduced from Eq. (4.15). The determination of $\mathcal{T}_{\rm eff}$ from the knowledge of $T_{\rm eff}$ is then possible [through Eq. (4.7)] if and only if the friction $\gamma(\omega)$ is known. This is indeed expected since, in the Langevin equation approach, the interaction between the Brownian particle and the surrounding medium is entirely contained in the friction.

In a simple power-law model, the aging properties of the medium can be encoded in a frequency-dependent effective temperature entirely defined by the exponent α and the characteristic frequency ω_0. Of course, these two variables may depend on the waiting time t_w, i.e., $\alpha=\alpha(t_w)$, $\omega_0=\omega_0(t_w)$. However, both α and ω_0 are considered as constants at the scale of the observation time which must then be kept small as compared with t_w. This condition can be written $\omega t_w\gg 1$, and states that the two time scales, respectively characterizing the times pertinent for the measuring process and the waiting time or the age of the system, must be kept well separated. The long-time limit in the present work should then not be regarded as the limit $(t-t')\to\infty$, but the limit $(t-t')\gg\omega_\delta^{-1},\omega_0^{-1}$, while keeping $(t-t')\ll t_w$. The investigation of the Brownian motion in this limit is sufficient to determine the two parameters α,ω_0 that characterize the aging of the medium under study (at given t_w) [13], in case the power-law model used in the model applies. Beyond this limit, the velocity correlation function C_{vv} for the Brownian particle cannot be put in a closed form, due to the violation of the regression theorem which prevents any simple description of C_{vv} in terms of Mittag-Leffler functions. On the other hand, it is possible to find a closed form for the correlation function of the bath coordinate (the noise), at least in the regime $\delta<1$ where this function is regular even without any cutoff at high frequency for the mode density of the bath. In this case, the effective temperature $T'_{\rm eff}(t-t')$ associated with the violation of the FDT for the aging medium in its time formulation satisfies the stability criterion $T'_{\text{eff}} > T$ at any time, so that our description of the aging of the medium is fully self-consistent.

The regular behavior of the noise correlation function is crucial to determine the effective temperature T'_{eff} of the bath. In particular, for the common case $\delta=1$, the white noise in Eq. (5.8) corresponding to the Einstein term has to be smoothed. The introduction of a Lorentzian cutoff function $f_c(\omega)=\omega_c^2/(\omega_c^2+\omega^2)$ in the definition of the friction in Eq. (4.8) gives [see, for instance, Eq. (4.8) in Ref. [2]]

$$\gamma(t-t') = \gamma_1 \omega_c e^{-\omega_c|t-t'|}. \tag{6.1}$$

In this expression, ω_c^{-1} acts as a measure of the memory time of the friction. Note that if $\gamma(t)$ is given by Eq. (6.1), then $\mathcal{T}_{\text{eff}}(\omega)$ is not given by Eq. (4.12) for $\omega > \omega_c$. However, since $\omega_0 \ll \omega_c$, $\mathcal{T}_{\text{eff}}(\omega) \simeq T$ at high frequency $\omega \simeq \omega_c$, so that $\mathcal{T}_{\text{eff}}(\omega)$ is not sensitive to the cutoff. A good approximation is then to keep Eq. (4.12) at any frequency, and consider the effect of the cutoff on the friction only. Note that $\gamma(t) \to 2\gamma_1 \delta(t)$ when $\omega_c \to \infty$. Equation (6.1) is actually more realistic than the Dirac peak which implies infinite variance of the force. The substitution $\delta(t-t')$ by $\gamma(t-t')/2$ with $\gamma(t-t')$ given by Eq. (6.1) is sufficient to smooth Eq. (5.8) and makes possible the determination of T_{eff} just at it has been reported above for $\delta < 1$. The noise correlation in Eq. (5.11) with $\gamma(t)$ given by Eq. (6.1) has then the same time dependence as in the case of two stochastic processes, which might be considered as realistic versions of almost uncorrelated noise, namely, the Ornstein-Uhlenbeck process and the random telegraph signal [24].

The out-of-equilibrium dynamics of the bath generate a time correlation of the noise force which satisfies a power law in time, after Eq. (5.8). This means that the slow relaxation of the out-of-equilibrium medium results in a correlation of the noise extending to a long-time scale. The out-of-equilibrium situation depicted in this work is thus very different from the introduction of a time-dependent temperature in Brownian thermal ratchets [25]. While $\langle F(t)F(t')\rangle \propto T\delta(t-t')$ at equilibrium, the nonequilibrium motion of the thermal ratchet is generated by a time-dependent temperature $T(t)$ in the noise correlation function, so that $\langle F(t)F(t')\rangle \propto T(t)\delta(t-t')$. In a few cases, the Dirac distribution is replaced by a peaked function such as Eq. (6.1), but it only shifts the noise properties from uncorrelated (white noise) to almost uncorrelated noise, in contrast with the long-range correlation found in the present work. Clearly, the out-of-equilibrium motion of the particle generated by the substitution $T\delta(t,t') \to T(t)\delta(t,t')$ in the physics of Brownian motors corresponds to a quite different situation where the Brownian particle either thermalizes instantaneously with different baths at different temperatures with which it is in contact at different times, or "follows" an adiabatic transformation of the bath which remains at equilibrium at any time.

Finally, we cannot expect that the power law envisioned for the friction in this work holds true at any frequency, and for any aging medium, beyond the fact that we have already outlined the condition $\omega \ll \omega_c$ with ω_c the characteristic cutoff frequency of the density spectrum of the medium under study (Drude, Debye, Fermi frequency, etc.). For instance, the Langevin equation with the quite different Coulomb friction has to be chosen if the random force in the medium originates from collision of molecules, or in granular materials [28]. We cannot either expect the power law for $\mathcal{T}_{\text{eff}}$ to be always valid. We can take advantage of the analogy formulated in this work between effective temperature and ac dielectric susceptibility to note that relaxation motion of internal modes in ferroelectrics gives a logarithmic contribution

$$\chi \propto [\ln(1/\omega\tau_0)]^{\xi} - i(\pi\xi/2)\text{sgn}(\omega)[\ln(1/\omega\tau_0)]^{\xi-1}],$$

the real part of which eventually dominates the real part of the susceptibility [18,26,27]. This logarithmic law might also apply to the effective temperature $\mathcal{T}_{\text{eff}}(\omega)$, instead of Eqs. (4.3) and (4.4), depending on the frequency range explored and the physics at the origin of the aging of the system investigated.

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APPENDIX A

In this appendix, we derive trivial analytic properties which we use to link the effective temperatures of the Brownian particle and the out-of-equilibrium medium with the velocity and random force correlation functions, respectively. Let $k(t)$ be a function of time and $\tilde{k}(t)=\theta(t)k(t)$ the causal function. The Fourier transform $\tilde{k}(\omega)$ of $\tilde{k}(t)$,

$$\tilde{k}(\omega) = \int_0^{\infty} e^{i\omega t} k(t) dt, \tag{A1}$$

is the convolution product of the Fourier transforms of $k(t)$ and $\theta(t)$, which can be written in the form of the well-known Kramers-Kronig relation:

$$\tilde{k}(\omega) = \frac{k(\omega)}{2} + i \int \frac{d\omega'}{2\pi} \frac{k(\omega')}{\omega - \omega'}. \tag{A2}$$

In addition, if $k(t)$ is even, its Fourier transform $k(\omega)$ is real so that

$$k(\omega) = 2\,\text{Re}\,\tilde{k}(\omega). \tag{A3}$$

As the velocity correlation function and the random force correlation function are even functions of time, Eqs. (A1) and (A3) apply. At equilibrium, they are Eqs. (3.1) and (3.2), respectively, for the velocity, and Eqs. (3.4) and (3.3), respectively, for the random force. Out of equilibrium, Eq. (A1) can be identified with Eq. (4.2) in which case Eq. (A3) takes the form

$$C_{FF}(\omega) = 2mk_B\,\text{Re}[\gamma(\omega)\mathcal{T}_{\text{eff}}(\omega)],$$

which is a combination of Eqs. (4.6) and (4.7). For the velocity, if Eq. (A1) is identified with Eq. (4.14), then Eq. (A3) takes the form

$$C_{vv}(\omega) = 2k_B \,\mathrm{Re}[\mu(\omega)\mathcal{T}_{\mathrm{eff}}^{(1)}(\omega)],$$

which is a combination of Eqs. (4.13) and (4.15).

APPENDIX B

Equation (4.7) determines $T_{\mathrm{eff}}(\omega)$ from the knowledge of $\mathcal{T}_{\mathrm{eff}}(\omega)$. Let us show that the opposite holds true, i.e., if we assume that T_{eff} is given by Eq. (4.11), Eq. (4.12) follows. According to Eqs. (4.7) and (4.11), $\mathcal{T}_{\mathrm{eff}}$ can be written as

$$\mathcal{T}_{\mathrm{eff}}(\omega) = T + T^{(2)}(\omega),$$

with

$$T(\omega/\omega_0)^{\alpha} = \mathrm{Re}\, T^{(2)}(\omega) - \frac{\mathrm{Im}\,\gamma(\omega)}{\mathrm{Re}\,\gamma(\omega)}\,\mathrm{Im}\, T^{(2)}(\omega). \tag{B1}$$

From Eq. (4.9), one has

$$\frac{\mathrm{Im}\,\gamma(\omega)}{\mathrm{Re}\,\gamma(\omega)} = \mathrm{sgn}[\omega]\cot\frac{\delta\pi}{2}, \tag{B2}$$

where $\mathrm{sgn}[\omega]$ denotes the sign of ω. After Eq. (B1), $T^{(2)} \propto \omega^{\alpha}$, and it has the same analytic properties as $\gamma(\omega)\propto\omega^{\delta-1}$ after Eq. (4.2). Therefore, we have just to make the substitution $\delta-1\to\alpha$, i.e., $\delta\to\alpha+1$ into Eq. (B2) to find

$$\frac{\mathrm{Im}\, T^{(2)}(\omega)}{\mathrm{Re}\, T^{\,2)}(\omega)} = -\,\mathrm{sgn}[\omega]\tan\frac{\alpha\pi}{2}. \tag{B3}$$

From Eqs. (B1) and (B3), it is then straightforward to find

$$\mathcal{T}_{\mathrm{eff}}(\omega) = T\left[1 + \frac{\sin(\delta\pi/2)}{\sin[(\delta+\alpha)\pi/2]}\left(\frac{-i\omega}{\omega_0}\right)^{\alpha}\right], \tag{B4}$$

which is Eq. (4.12). The function $\mathcal{T}_{\mathrm{eff}}(\omega)$ is analytic in the whole complex plane, except for a cut on the real negative axis. Note also that, for real ω,

$$\mathrm{Re}\,\mathcal{T}_{\mathrm{eff}}^{(2)}(\omega) = T\left[1 + \left(\frac{|\omega|}{\omega_0'}\right)^{\alpha}\right], \tag{B5}$$

with

$$\omega_0' = \omega_0\left[\frac{\sin(\delta\pi/)\cos(\alpha\pi/2)}{\sin[(\delta+\alpha)\pi/2]}\right]^{-1/\alpha}. \tag{B6}$$

$\mathcal{T}_{\mathrm{eff}}^{(2)}$ can then be viewed as an analytical continuation in the complex frequency plane of an effective temperature which takes the form of Eq. (4.11), with renormalized ω_0 parameter according to Eq. (B6). The functional dependence on frequency is then the same in both cases, and the aging system is then consistently described by T_{eff} and (or) $\mathcal{T}_{\mathrm{eff}}$ for the particular choice of a power law for the friction.

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